One physical problem with possible mathematical significance

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Dedicated to Professor I.M. Gelfand
on his 75th birthday

Abstract. *We analyze three-dimensional gauge theory with Chern-Simons action, which arises in the context of high T_c superconductivity. We show that the main effect in this theory is the change of spin and statistics of charged particles. Also, the amplitudes of particle propagation acquire phases dependent on the topology of the knot formed by their trajectories. In the end, we discuss possible generalizations of this theory, and their relations to strings.*

I will describe in this short paper a physical problem, which seems to me mathematically meaningful as well.

It arises from consideration high T_c superconductors. After cleansing it of many physical details it can be formulated as follows.

Let us consider a charged particle which sweeps a closed loop Γ in the three dimensional space-time. It interacts with an abelian gauge field, described by one-form:

$$A = A_\mu \mathrm{d}\, x^\mu \tag{1}$$

It is assumed that the action, describing this field, is given by Chern-Simons term (such

Key-Words: Chern-Simons action, knots, spin and statistic, super condutivity strings.
1980 MSC: 81 G, 82 A 15.

actions where first considered by J. Schoenfeld and by R. Jackiw and S. Templeon):

$$S(A) = \frac{\vartheta}{16\pi^2}\int_M A \wedge \mathrm{d}A \tag{2}$$

where M is a three-dimensional manifold in consideration. The particle acquires a loop dependent phase factor which contains most of physical information and is defined by:

$$\phi(\Gamma) = \langle e^{i\oint_\Gamma A}\rangle_A \underset{\text{def}}{=} \int [\mathcal{D}A] e^{i\int(A)} \exp\{i\oint_\Gamma A\} \tag{3}$$

Here $[\mathcal{D}A]$ means functional integration with respect to gauge orbits of the field.

The action (2) is remarkable. It doesn't depend on the metric tensor on the manifold M, and in this sense is topologically invariant. One can expect that the partition function:

$$Z[M] = \int \mathcal{D}A \exp\{i\int(A)\} \tag{4}$$

must also be topologically invariant. This is indeed the case, as was shown by A. S. Schwartz and this invariaht is precisely Ray-Singer torsion.

If one continues this line of arguments, an impression arises, that $\phi(\Gamma)$ is also topologically invariant and should remain unchanged under small deformations of Γ ([1]). This is not exactly so, because of interesting quantum effects, as we will show now. However $\phi(\Gamma)$ still have certain topological meaning, depending on a knot formed by Γ.

It is easy to compute the functional integral in (3), since the action (2) is quadratic. The answer is given by:

$$\left\langle \exp\{i\oint_\Gamma A\}\right\rangle_A = \exp\{\frac{1}{\vartheta}\int_{\Gamma\times\Gamma} \epsilon_{\mu\nu\lambda}\left(\frac{x^\lambda - y^\lambda}{(x-y)^3}\right) \mathrm{d}x^\mu \mathrm{d}y^\nu\} \tag{4}$$

According to the argument given above this expression must be «almost» topological invariant. By that we mean the following. While the action (2) is invariant under diffeomorphisms of M, we have to regularize it in order to make the functional integral well defined. This is achieved by adding to (2) terms with higher derivatives of A. The simplest regulator is:

$$S_{\text{reg}} = \Lambda^{-1}\int_M F \wedge^* F \tag{5}$$

(1) E. Witten (private communication through P. Nelson) conjectured that these invariants can be related to Jones polynomials. Similar suggestions was made to me by I. M. Singer.

where Λ is a cut-off (which we must tend to ∞ at the end) and $F = \mathrm{d}\,A$. Now, the regulator depends on the metric on M. That implies that our answers, obtained by taking $\Lambda \to \infty$ may depend on the metric, but only locally. We come to the conclusion that must be topologically invariant modulo local terms depending on Γ. This general argument is confirmed by the explicit computation of (4). We can transform this expression as following:

$$
\begin{aligned}
I &\equiv \frac{1}{4\pi}\int_{\Gamma\times\Gamma}\epsilon_{\mu\nu\lambda}\mathrm{d}\,x^{\mu}\mathrm{d}\,y^{\nu}\partial_{\lambda}\,\frac{1}{|x-y|} = \\
&= \int_{\Sigma_{\Gamma}}\epsilon_{\mu\nu\lambda}\mathrm{d}\,x^{\mu}\wedge\mathrm{d}\,x^{\nu}\oint_{\Gamma}\mathrm{d}\,y^{a}\delta(x-y)
\end{aligned}
\tag{6}
$$

(where Σ_1 is some surface bounded by Γ.)

In order to make (6) well defined we have to remember that due to regulators (5) the δ-functions in (6) is smeared. Precise way of doing this is irrelevant. It suffices to take:

$$
\delta(x-y) = \frac{1}{(2\pi\epsilon)^{3/2}}e^{-\frac{(x-y)^2}{2\epsilon}}\epsilon\infty\Lambda^{-1}
\tag{7}
$$

When we substitute (7) into (6), we get two types of contributions. First of all, the δ-functions in (6) «clicks» whenever the loop Γ' intersects the surface Σ_{Γ}. If the knot, formed by is trivial, we can choose: Σ_{Γ} in such a way, that intersections are absent. Let us examine this case first. The only contribution to the integral (6) then comes from the thin strip (the thickness the order of ϵ) attached to the boundary of Σ_{Γ}. Using explicit parametrization of this strip, we obtain in the limit $\epsilon \to 0$ the following formula:

$$
I = \frac{1}{2\pi}\oint_{\Gamma}\mathrm{d}\,\vec{x}[\vec{n}\times\frac{\mathrm{d}\,\vec{n}}{dfs}]
\tag{8}
$$

where $\vec{n}(s)$ is a normal to Γ, tangent to Σ_{Γ}. Of course, the answer must be independent from the choice of Σ_{Γ}, and indeed, (8) can be cast in the form in which this independence is explicit. It can be rewritten as a multivalued functional (analogous to the ones considered by Novikov and Witten in different context):

$$
I = \frac{1}{2\pi}\int_0^L\mathrm{d}\,s\int_0^1\mathrm{d}\,u\vec{e}(s,u)[\partial_s\vec{e}\times\partial_u\vec{e}]
\tag{9}
$$

where

$$
\vec{e}(s,u) = \begin{cases}\vec{e}(s) & u = 1\\ \text{const} & u = 0\end{cases}
$$

and $\vec{e}(s)$ is a tangent vector to Γ. The integral (9) is independent modulo integer of the choice of interpolation to $\vec{(}s)$. Before discussing consequences of this computation, we have to examine the case of non-trivial knots. For that we have to analyse what happens to the integral I when we deform Γ in such a way that it crosses itself. An interesting fact about it is that while the double integral in (4) converges even when the loop Γ is self-intersecting, it is discontinuous, when under the change of Γ, it crosses itself. It is easy to check by explicit computation that

$$I(\times) = I(\times) + 1 \qquad (10)$$
$$I(\times) = I(\times) - 1$$

Now, we see that using this rule we can find the integer part of I for a given knot. The general structure of I is thus the same of two terms: one is continuous functional of the curve (8), while the other is integer topological invariant. This structure resembles the one of spectral asymmetry of Dirac operator in odd-dimensions, which contains continuous part coinciding with the Chern-Simons term and also experiences integer jumps under the change of gauge fields.

When $\vartheta = \frac{1}{2}\pi s$ and s is integer or half-integer, all these integer discontinuites do not contribute to the phase factor $\phi(\Gamma)$ and in this case the expression (9) can be viewed as a well defined action for the dynamical variable $\vec{e}(s)$. This action gives rise to the Poisson brackets, defined in a standard way by the canonical 2-form. Simple computation shows, that in the case of (9) these brackets have the form:

$$[e_\alpha, e_\beta] = \epsilon_{\alpha\beta\gamma} e_\gamma \qquad (11)$$

It is not surprising, therefore, that quantization of the particle, carrying extra factor $\phi(\Gamma)$, leads to the $2s+1$-dimensional representation of (11) and the particle, which was spinless at the beginning, acquires spin s due to its interaction with A-fields.

Let us recapitulate. We started from the spinless particle «dressed» by the long-ranged A-field. After accounting if we got spin s particle without long-range interaction. The point-like interaction is present for half-integer interaction. The point-like interaction is present for half-integer spins, since, according to (10) we get negative sign at the moment of self-crossing of the trajectory. Thus, topological action (2) changes spin and statistics of particles.

Where s is not integer or half-integer, then the interaction described by (10) has infinite range, since when two pieces of trajectory cross each other, the factor $\phi(\Gamma)$ acquire a phase $e^{\mu\pi i s}$ which stays forever. Effect of this interaction is to renormalize s. Perhaps, it will tend to some fixed point. At present, only the leading term of the β-function for small s is known.

An obvious generalization of the above problem is to consider non-abelian analogue of it. In this case A takes values in the Lie algebra of some compact group G. The action (2) is replaced by:

$$S(A) = \frac{\vartheta}{16\pi^2} Tr \int (A \wedge dA + A \wedge A \wedge A) \qquad (12)$$

It is well known, that in order to maintain gauge invariance of (12) (the gauge transformation is given by:)

$$A \Rightarrow g^{-1}Ag + g^{-1}\mathrm{d}\,g$$

the values of ϑ must be quantized:

$$\vartheta = 2\pi k;\ k\text{-integer} \tag{13}$$

We can also introduce an obvious, gauge invariant, generalization of the phase factor (3):

$$\begin{aligned} \phi(\Gamma) &= \langle Tr\psi(\Gamma)\rangle \\ \psi(\Gamma) &= P\exp\oint_{\Gamma} A \end{aligned} \tag{14}$$

(here P-means ordering of the exponential in (14)). The same argument as before show that $\phi(\Gamma)$ must be «almost» topological invariant of the knot. It is not clear at present, however, what is the explicit expression for $\phi(\Gamma)$ and what kind of physical effect it has on test particles in the non-abelian case. I can add only few disjoint pieces of information, concerning this question. First of all, if the group $G = SU(N)$, then for $N \to \infty$ limit one can find a close equation in the loop space for the quantity $\phi(\Gamma)$. If Γ is parametrized by $x^{\mu} = x^{\mu}(s)$, then the equation has the form:

$$\begin{aligned} K\frac{\delta\phi(\Gamma)}{\delta x^{\mu}(s)} = \epsilon_{\mu\nu\lambda}\int_0^4 \mathrm{d}\,u(\dot{x}^{\nu}(s)\dot{x}^{\lambda}(u))\cdot \\ \cdot\,\delta(x(s)-x(u))\,\phi(\bar{\Gamma})\ \phi(\bar{\bar{\Gamma}}) \end{aligned} \tag{15}$$

where, the loop Γ has self-intersection due to the δ-function and $\bar{\Gamma}$ and $\bar{\bar{\Gamma}}$ are two part of Γ. As in the ordinary gauge theory, one can reproduce perturbation expansion in $\frac{1}{k}$ by iterating (15). The way of deriving (15) is identical to the one in ordinary gauge theory, while the resulting equation is much simpler. It would be very interesting to analyze its solution non-perturbatively.

Another piece of information deals with non-abelian generalization of «crossing» rules (10).

Suppose that Γ_1 and Γ_2 are two small pieces of the loop. In the abelian case we can rewrite (10) in the following way. If Γ_1 goes over Γ_2 we shall denote this fact by writing their contribution to the phase factor as $\psi(\Gamma_1)\psi(\Gamma_2)$, in the opposite case ($\Gamma_1$ crosses under Γ_2) we shall write it as $\psi(\Gamma_2)\psi(\Gamma_1)$. In these notations, the rules (10) can be summarized by the equation:

$$\psi(\Gamma_1)\psi(\Gamma_2) = e^{\mu\pi i s}\psi(\Gamma_2)\psi(\Gamma_1) \tag{16}$$

Perhaps, the non-abelian generalization of it would be:

$$\psi(\Gamma_1) \otimes \psi(\Gamma_2) = S(\psi(\Gamma_2) \otimes \psi(\Gamma_1)) \tag{17}$$

where $\psi(\Gamma_{1,2})$ are elements of the group G, while S is an «S-matrix» mapping $G \otimes G$ to $G \otimes G$. It is well known, that the structure (17) requires that S must satisfy Yang-Baxter relations. Perhaps, developing this line of arguments, it will be possible to verify Witten-Singer conjecture, which I mentioned at the beginning.

The appearence of the integrable S-matrix and its explicit form can be understood also from the different point of view. Let us try to compute the phase factors using the gauge $A_0 = 0$. Then the field A is Gaussian and the propagator takes the form:

$$\langle A_n(x) A_m(y) \rangle = sqn(x^0 - y^0)\epsilon_{nm}\delta^{(2)}(x-y)(n, m = 1,2) \tag{18}$$

That implies that the only contribution to the phase factor comes from the intersection points $x = y$ of the projection of the knot on the plane $x^0 = \text{const}$. Moreover, only interaction between different branches must be accounted. That gives S-matrix :

$$S_{ij,kl} = \left\langle (P\exp\int_{\Gamma_1} A_n \mathrm{d}x^n)_{ij} (P\exp\int_{\Gamma_2} A_m \mathrm{d}y^m)_{kl} \right\rangle \tag{19}$$

for which explicit formula can be given (we shall do it elsewhere). After computing (19), the phase factor for the knot is obtained by taking «partition function of the knot» i. e. by multiplying S matrices, corresponding to all intersections in the projection of the knot. So we conclude, that:

$$\phi(\Gamma) = \left\langle P\exp\oint_{\Gamma} A_n \mathrm{d}x^n \right\rangle_{\{A\}} \propto$$

$\propto$ partition function of the knot Γ.

There are many possible generalizations for the structures which I have described. One can consider higher dimensional manifolds and change paths to hypersurfaces. This is possible for any odd dimensionality, by changing one-form A to the n-form. For even dimensionality one can take Chern classes instead of Chern-Simons form to the action. This possibility has been considered by L. Baulien and I. Singer, but consequences for the phase factors are still to be analyzed.

It is conceivable, that one of these theories will be equivalent to string theory and will contain theory of gravity. The mechanism for appearence of strings may be usual collimation of color-elecric fluxes. The gravitons may appear due to the change of spins of fundamental fields - it is easy to imagine that spin one field A being charged acquires spin two and becomes a graviton.

It would be very satisfying if the same mechanism is responsible for fundamental structure of space-time, high T_c-superconductivity and last but not least for interesting mathematics.

Manuscript received: September 20, 1988